Nonparametric Techniques in Statistical Inference

Edited by

MADAN LAL PURI

Cambridge University Press

PITMAN EFFICIENCIES OF TESTS BASED ON SPACINGS

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1 INTRODUCTION

Let $X_1, X_2, \ldots, X_{n-1}$ be $(n-1)$ independently and identically distributed random variables $(n > 2)$ with a common distribution function (df) $G(x)$. In this paper we are concerned with the Pitman efficiencies of tests based on spacings (see (1) below) in the goodness-of-fit problem, i.e. the problem of testing $G(x)$ is equal to a specified continuous df $G_0(x)$. In this case, the simple probability transformation on the random variables would permit us to equate $G_0(x)$ with the uniform df on [0, 1]. We assume that this is done, and so the null hypothesis states $G(x) =$ the uniform df on [0, 1].

Let $X'_1 \leqslant X'_2 \leqslant \ldots \leqslant X'_{(n-1)}$ be the order statistics. The sample spacings $\{D_1, D_2, \ldots, D_n\}$ are defined by

$$D_i = X'_i - X'_{i-1} \quad (i = 1, \ldots, n), \tag{1}$$

where we put $X'_0 = 0$, $X'_n = 1$.

Some of the spacings tests for the goodness-of-fit proposed in the literature are those based on $V_r(n)$, $U(n)$ and $L(n)$ defined below. In each case the hypothesis is rejected when the absolute value of the statistic is large. The statistics

$$V_r(n) = \sum_1^n (nD_i)^r/n \quad (r > -\tfrac{1}{2})$$

were suggested by Kimball (1950)—of which the statistic suggested by Greenwood (1946) is the case with $r = 2$. Kendall in the discussion on Greenwood (1946) suggested the statistic

$$U(n) = \sum_1^n |nD_i - 1|/n$$

which was later studied by Sherman (1950). Darling (1953) proposed the statistic

$$L(n) = \sum_1^n \log (nD_i)/n.$$

In this paper we present a unified treatment of the computation of Pitman (asymptotic relative) efficiencies of such tests. Though it is known that the efficiency of any test symmetric in the spacings is zero relative to the Kolmogorov–Smirnov test (as shown for instance by Cibisov (1961)) it would be useful to know about the efficiency of one symmetric spacings test relative to another. Table 1 in § 3 gives the efficacies of the tests based on $V_r(n)$, $U(n)$ and $L(n)$ from which the relative efficiencies can be computed. Finally it is shown that among a large class of symmetric tests based on spacings the test based on $V_2(n)$ has maximum efficacy.

We now describe the alternative hypothesis. Being concerned with Pitman efficiencies, we will specify the alternative hypothesis by a df depending on n and converging to the null hypothesis. Under the alternative hypothesis,

$$G_n(x) = x + L_n(x)/n^{\delta} \quad (x \in [0,1]), \tag{2}$$

where $L_n(0) = L_n(1) = 0$ and $\delta \geqslant \frac{1}{4}$. We further assume that $L_n(x)$ is twice differentiable on $[0, 1]$ and there is a function $L(x)$ which is twice continuously differentiable and

$$L(0) = L(1) = 0,$$

$$n^{\delta^*} \sup_{0 \leqslant x \leqslant 1} |L_n(x) - L(x)| = o(1),$$

$$n^{\delta^*} \sup_{0 \leqslant x \leqslant 1} |L'_n(x) - l(x)| = o(1),$$

$$n^{\delta^*} \sup_{0 \leqslant x \leqslant 1} |L''_n(x) - l'(x)| = o(1),$$

where $l(x)$ and $l'(x)$ are the first and second derivatives of $L(x)$ and $\delta^* = \max(0, \frac{1}{2} - \delta)$.

These sequence of alternatives are 'smooth' in a certain sense and have been considered before in this problem, for instance, see Cibisov (1961), Weiss (1965). These alternatives have carrier $[0, 1]$ just as the hypothesis. (The carrier of a distribution is the smallest closed set with probability 1.) This seems to be a natural condition on the alternatives if the test is to be based on spacings since the spacings, as defined in (1), do not make allowances for sub-intervals of $[0, 1]$ with probability zero and the intervals $(-\infty, 0)$ and $(1, \infty)$, if they have positive probability. Weiss (1962) has constructed an interesting example of a sequence of alternatives with non-constant carrier having limiting power 1 under a spacings test but which have limiting power equal to test size under the Kolmogorov–Smirnov test. We discuss this example again in § 3.

Previous approaches to the computation of Pitman efficiencies mostly used fixed alternatives. In this case, the limiting distributions of test statistics based on spacings become difficult to obtain. Some special cases have been treated in Weiss (1957), Pyke (1965), Jackson (1967). This still does not give Pitman efficiencies since the usual method of differentiating the asymptotic mean, etc., cannot in general be justified. Cibisov (1961) and Weiss (1965) met with success using alternatives of the form (2). Theorems 1 and 2 in § 2 give the limiting distribution of the empirical df of the 'normalized' and 'modified' spacings, where these terms are defined. In § 3 we show how to obtain Pitman efficiencies of tests based on spacings, including many of those proposed in the literature. One can also consider the problem of testing for uniformity of a distribution on a circle. In § 3 we remark that the asymptotic theory of circular spacing tests is the same as that of linear spacing tests. Thus Pitman efficiencies of circular spacings tests are the same as those of the corresponding linear spacings tests.

2 LIMITING DISTRIBUTION OF THE EMPIRICAL DF OF 'NORMALIZED' AND 'MODIFIED' SPACINGS

Under the null hypothesis, $E(D_i) = 1/n$ for all i. We will therefore call $\{nD_i, i = 1, \ldots, n\}$ 'normalized' spacings. Let

$$\{h_{n1}, h_{n2}, \ldots, h_{nn}\} \quad (n = 2, 3, \ldots),$$

be a triangular array of positive numbers. We shall call

$$\{nD_1/h_{n1}, nD_2/h_{n2}, \ldots, nD_n/h_{nn}\}$$

'modified' or 'adjusted' spacings. For example, one way of adjusting the spacings is to divide them by their expectations under some arbitrary df. This might help to enlarge the class of statistics based on spacings and may be likened to the method of using Normal scores and other scores in rank statistics.

We now define the empirical df's $H_n(x)$ and $H_n^*(x)$ of the normalized and modified spacings, respectively. Let

$$H_n(x) = \sum_1^n I(nD_i; x)/n \quad (x \geqslant 0) \tag{3}$$

and

$$H_n^*(x) = \sum_1^n I(nD_i/h_{ni}; x)/n \quad (x \geqslant 0), \tag{4}$$

where

$$I(z; x) = \begin{Bmatrix} 1 & z \leqslant x \\ 0 & z > x \end{Bmatrix}. \tag{5}$$

When we deal with modified spacings we shall always assume that there exists a continuous function $h(p)$ on $(0, 1)$ such that

$$\max_{1 \leqslant i \leqslant n} \sqrt{n}\, |h(i/(n+1)) - h_{ni}| = o(1)$$

and that, for each x,

$$\int_0^1 (1-e^{-xh(p)})\,dp < \infty, \quad \int_0^1 e^{-xh(p)} l(p)\,h(p)\,dp < \infty,$$

$$\int_0^1 L(p)\,l'(p)\,h(p)\,e^{-xh(p)}\,dp < \infty \quad \text{and} \quad \int_0^1 h^2(p)\,l^2(p)\,e^{-xh(p)}\,dp < \infty.$$

Call this condition (M).

Define

$$\zeta_n(x) = \sqrt{n}(H_n(x) - F_n(x)) \quad (x \geqslant 0) \tag{6}$$

and

$$\zeta_n^*(x) = \sqrt{n}(H_n^*(x) - F_n^*(x)) \quad (x \geqslant 0), \tag{7}$$

where

$$F_n(x) = 1 - e^{-x} + \left(\int_0^1 l^2(p)\,dp\right) e^{-x}(x - x^2/2)/n^{2\delta} \tag{8}$$

and

$$F_n^*(x) = \int_0^1 (1-e^{-xh(p)})\,dp + \left(\int_0^1 x\,e^{-xh(p)} l(p)\,h(p)\,dp\right)\Big/ n^{\delta}$$
$$+ \left(\int_0^1 [-xL(p)\,l'(p)\,h(p)\,e^{-xh(p)} - x^2 h^2(p)\,l^2(p)\,e^{-xh(p)}/2]\,dp\right)\Big/ n^{2\delta}. \tag{9}$$

Note that $\lim_{x\to\infty} \zeta_n(x) = 0$ and $\lim_{x\to\infty} \zeta_n^*(x) = 0.$

We therefore put $\zeta_n(\infty) = \zeta_n^*(\infty) = 0$. Then, it is easy to see that

$$\{\zeta_n(x), 0 \leqslant x \leqslant \infty\} \quad \text{and} \quad \{\zeta_n^*(x), 0 \leqslant x \leqslant \infty\}$$

are measurable and are stochastic processes in $D[0,\infty]$ endowed with the Skorohod topology. The original Skorohod topology was for $D[0,1]$, but is applicable to any $D(K)$, when there is a strictly increasing continuous transformation of $[0,1]$ onto K.

We now state two theorems.

Theorem 1. *Under the alternatives in* (2), *the processes* $\{\zeta_n(x), 0 \leqslant x \leqslant \infty\}$ *converge weakly to a Gaussian process* $\{\zeta(x), 0 \leqslant x \leqslant \infty\}$ *in* $D[0,\infty]$ *having mean function zero and covariance function*

$$K(x,y) = e^{-y}(1 - e^{-x} - xy\,e^{-x}) \quad \text{for } 0 \leqslant x \leqslant y \leqslant \infty. \tag{10}$$

Theorem 2. *Under the alternatives* (2) *and under condition* (M), *the process* $\{\zeta_n^*(x), 0 \leqslant x \leqslant \infty\}$ *converges weakly to the Gaussian process*

$$\{\zeta^*(x), 0 \leqslant x \leqslant \infty\}$$

in $D[0,\infty]$ *with mean function zero and covariance function*

$$K^*(x,y) = \int_0^1 e^{-yh(p)}(1-e^{-xh(p)})\,dp$$
$$- xy\left(\int_0^1 h(p)\,e^{-xh(p)}\,dp\right)\left(\int_0^1 h(p)\,e^{-yh(p)}\,dp\right) \quad \textit{for } 0 \leqslant x \leqslant y \leqslant \infty. \tag{11}$$

Theorem 2 implies Theorem 1. Theorem 2 is no more difficult to prove than Theorem 1. The proof of these theorems are involved and will not be given here. They use the well-known transformation from general spacings to the uniform spacings and from the uniform spacings to ratios of exponential random variables to their sum. See for instance Pyke (1965), Weiss (1962). After this stage, we have to deal with the empirical distribution function of random variables with random perturbations and a random scale factor, i.e. with something like

$$\sum_1^n I(W_i/\theta_{ni}\, W_n^*;\, x)/n, \tag{12}$$

where $W_1, \ldots, W_n$ are independently and identically distributed random variables, θ_{ni} are random variables approximately equal to $\theta(i/(n+1))$, where $\theta(p)$ is a well behaved function and W_n^* is a random variable with asymptotic normal distribution with mean 1 and variance $1/\sqrt{n}$. In a subsequent paper, Sethuraman and J. S. Rao (1969), we establish weak convergence of such empirical df's under more general conditions than required here. Since it seems to us that such results are of sufficient independent interest, we have relegated the proof of Theorems 1 and 2 to our forthcoming paper.

Theorems 1 and 2 allow us to obtain the limiting distributions of a host of functionals of $\zeta_n(x)$ and $\zeta_n^*(x)$, by just invoking the invariance principle. We do this in the next section and compute the Pitman efficiencies of several tests.

3 PITMAN EFFICIENCIES OF TESTS BASED ON SPACINGS

The Pitman asymptotic relative efficiency (ARE) of a test relative to another test is defined to be the limit of the inverse ratio of sample sizes required to obtain the same limiting power at a sequence of alternatives converging to the null hypothesis. This limiting power should be a value between the limiting test size, α, and the maximum power, 1. If the limiting power of a test at a sequence of alternatives is α, then its ARE with respect to any other test with the same test size and with limiting power greater than α, is zero. On the other hand, if the limiting power of a test at a sequence of alternatives converges to a number in the open interval $(\alpha, 1)$, then a measure of rate of convergence, called 'efficacy' can be computed. Under certain standard regularity assumptions (see e.g. Fraser (1957)), which include a condition about the nature of the alternative, asymptotic normal distribution of the test statistic under the sequence of alternatives, etc., this efficacy is given by

$$\text{efficacy} = \mu^2/\sigma^2. \tag{13}$$

Here μ and σ^2 are the mean and variance of the limiting normal distribution under the sequence of alternatives when the test statistic has been normalized to have a limiting standard normal distribution under the hypothesis. In such a situation, the ARE of one test with respect to another is simply the ratio of their efficacies.

We first consider tests statistics based on the normalized spacings. Let $m(x)$ be a function on $(0, \infty)$ and let

$$T_n = \sum_1^n m(nD_i)/n = \int_0^\infty m(x)\, dH_n(x). \tag{14}$$

Let $m(x)$ be absolutely continuous and of bounded variation in the interval $(\epsilon, 1/\epsilon)$ for each $\epsilon > 0$, and let

$$\int_0^1 m'(x) \sqrt{((1-e^{-x}) \log\log (1-e^{-x})^{-1})}\, dx < \infty \tag{15}$$

and
$$\int_0^\infty m'(x) \sqrt{(e^{-x} \log x)}\, dx < \infty. \tag{16}$$

Further let
$$\int_0^\infty m(x)\, dF_n(x) < \infty \tag{17}$$

and
$$\int_0^\infty \int_0^\infty m'(x)\, m'(y)\, K(x,y)\, dx\, dy < \infty, \tag{18}$$

where $F_n(x)$ and $K(x,y)$ are as in (8) and (10).

Conditions (15) through (18) are sufficient conditions for

$$\int_0^\infty m(x)\, dy\,(x),\ y(x) \in D$$

to be á functional on $D[0, \infty]$ which is continuous with probability 1 under $\{\zeta(x), 0 \leqslant x \leqslant \infty\}$. Conditions (15) and (16) are smoothness conditions on $m(x)$ based on the growth rate of $\zeta(x)$ at $x = 0$ and $x = \infty$ given by the law of the iterated logarithm while (17) and (18) make the mean and variance of T_n finite. Some examples of functions $m(x)$ satisfying these conditions are

$$\begin{aligned} m(x) &= x^r \quad (r > -\tfrac{1}{2}), \\ m(x) &= |x-1|/2, \\ m(x) &= \log x. \end{aligned} \tag{19}$$

T_n will be called a regular spacings statistic when $m(x)$ satisfies (15) through (18).

The following is easily deduced from Theorem 1.

Theorem 3. *Let* $m(x)$ *satisfy conditions* (15) *through* (18). *Let*

$$S_n = \sqrt{n}\left(\int_0^\infty m(x)\,dH_n(x) - \int_0^\infty m(x)\,dF_n(x)\right). \tag{20}$$

Under the sequence of alternatives (2), S_n *has a limiting normal distribution with mean* 0 *and variance*

$$\int_0^\infty \int_0^\infty m'(x)\,m'(y)\,K(x,y)\,dx\,dy < \infty.$$

Conditions (15) through (18) can be relaxed in Theorem 3. However, since the test statistics that we deal with are covered by the examples of (19) we will not try to do this here.

By expanding the mean $\int_0^\infty m(x)\,dF_n(x)$ of T_n under the null and alternative hypothesis we readily obtain that

$$\text{efficacy of } T_n = \begin{cases} 0 & \text{if } \delta > \tfrac{1}{4}, \\ \dfrac{\left(\int_0^1 l^2(p)\,dp\right)^2 \left(\int_0^\infty m'(x)\,e^{-x}(x - x^2/2)\,dx\right)^2}{\int_0^\infty \int_0^\infty m'(x)\,m'(y)\,K(x,y)\,dx\,dy} & \text{if } \delta = \tfrac{1}{4}. \end{cases} \tag{21}$$

It thus turns out that tests which are symmetric in the sample spacings cannot discriminate alternatives if they are at a distance of order $n^{-\delta}$ from the hypothesis and $\delta > \frac{1}{4}$. This is a disturbing feature. It was first demonstrated by Cibisov (1961) where he computed the likelihood ratio of the ordered sample spacings. However, we know that there exist tests which can discriminate alternatives at a distance of order $n^{-\frac{1}{2}}$ from the hypothesis. The Kolmogorov–Smirnov test is an example. This poor showing of tests symmetrically based on the spacings may be somewhat explained as follows. The sample spacings form a sufficient statistic since they are equivalent to the order statistics. Under the null hypothesis they form an interchangeable collection of random variables. However, limiting oneself to symmetric functions of the spacings entails loss of information, since the original order statistics cannot be recovered now.

Formula (21) enables us to compute the efficacies of tests symmetrically based on spacings. We present below a table giving the efficacies of the tests $V_r(n)$, $L(n)$ and $U(n)$.

Table 1. *Efficacies of some tests based on spacings*

Test statistic	Efficacy $\Big/ \left(\int_0^1 l^2(p)\,dp\right)^2$
$V_r(n): r = 0\cdot0$	0·0000
$r = 0\cdot5$	0·6760
$r = 1\cdot0$	0·0000
$r = 1\cdot5$	0·9700
$r = 2\cdot0$	1·0000
$r = 2\cdot5$	0·9728
$r = 3\cdot0$	0·9000
$r = 3\cdot5$	0·7976
$r = 4\cdot0$	0·6792
$U(n)$:	0·5726
$L(n)$:	0·3876

It is interesting to note that the efficacies above depend on the alternative only through the multiplying constant, $\left(\int_0^1 l^2(p)\,dp\right)^2$. Thus the efficiencies become independent of the alternative hypothesis. The table also indicates that the Greenwood statistic $V_2(n)$ has maximum efficacy in the limited comparison that was made. Let $T_n = \int_0^\infty m(x)\,dH_n(x)$ be a regular spacings test. It has maximum efficacy among all regular spacings test if and only if

$$\frac{\left(\int_0^\infty m'(x)\,e^{-x}(x-x^2/2)\,dx\right)^2}{\int_0^\infty \int_0^\infty m'(x)\,m'(y)\,K(x,y)\,dx\,dy}$$

is a maximum. This condition is equivalent to the condition that there is a constant λ such that

$$\int_0^\infty m'(y)\,K(x,y)\,dy = \lambda e^{-x}(x-x^2/2).$$

With $m(x) = x^2$, $m'(x) = 2x$, it is easy to verify the above condition. Thus the Greenwood statistic $V_2(n)$ has maximum efficacy among all regular spacings statistics.

Using Theorem 2 we can give the efficacies of tests of the form $\sum_1^n m(nD_i/h_{ni})/n$. We shall not give any general theorems on and computation of efficacies of such tests symmetrically based on modified spacings, since it is not clear which kind of modification to choose. However, we will broadly indicate what possibilities occur when one uses modified spacings. Put $h_{ni} = n/(n-i+1)$. Then the modified spacings become $\{nD_1, (n-1)D_2, \ldots, D_n\}$. Consider a test based on the mean, $M(n)$, of these modified spacings. In large samples, it is essentially a test based on the mean of $X_1, \ldots, X_n$. Either using Theorem 2, or directly, we can show that the efficacy of $M(n)$ is

$$\begin{array}{lll} 12\left(\int_0^1 l(p)\,p\,dp\right)^2 & \text{if} \quad \int_0^1 pl(p)\,dp \neq 0 & \text{and} \quad \delta = \tfrac{1}{2}, \\ 0 & \text{if} \quad \int_0^1 pl(p)\,dp = 0 & \text{and} \quad \delta = \tfrac{1}{4}. \end{array}$$

Thus, this test has efficiency ∞, at a sequence of alternatives with $\delta = \frac{1}{4}$, relative to a test based symmetrically on the spacings, say $V_2(n)$, when $\int_0^1 pl(p)\,dp \neq 0$, but has efficiency 0 when $\int_0^1 pl(p)\,dp = 0$. Thus at least for this test $M(n)$, there is a sequence of alternatives at which it does not fare as well as tests based symmetrically on the spacings. Whether this is a general phenomenon or not is not known. It would be interesting to investigate this problem.

It is known that the Kolmogorov–Smirnov statistic discriminates alternatives of the form (2) with $\delta = \frac{1}{2}$ or more generally, alternatives with df $G_n(x)$ with $\sqrt{n}\sup_x |G_n(x)-x| \to a$ constant $\neq 0$. Thus the Kolmogorov–Smirnov test has efficiency ∞ with respect to any test which is symmetric in the spacings. (See for instance Cibisov (1961).) Weiss (1962) has given an example of a sequence of alternatives, namely the uniform distribution on $\left[\frac{1}{\log n}, 1\right]$, at which the Kolmogorov–Smirnov test has efficiency 0 compared to the symmetric spacings test based on

$$D(n) = \max_i D_i.$$

The alternatives here do not have a constant carrier and the definition of the first sample spacing D_1 as X_1' inflates it considerably when the alternative is true. This accounts for the better performance of the test based on $D(n)$. As we described in § 1, we feel that all distributions considered must have the same carrier if spacings are to retain their meaning and do not get inflated as in this case.

The results established here apply with equal force to the goodness-of-fit problems on the circle. Consider n random variables $X_1, X_2, \ldots, X_n$ distributed independently and identically on a circle with unit circumference. The null hypothesis is one which states that the distribution is uniform on the circle. Ordering the observations as $X_1' \leqslant X_2' \leqslant \ldots \leqslant X_n'$, the sample spacings ($n$ in number), which are the arc-lengths between the successive sample observations, may be defined by

$$D_i = X_i' - X_{i-1}' \quad (i = 1, \ldots, n),$$

where we put $X_0' = X_n' - 1$. Under the null hypothesis, the distribution of these spacings is the same as those from a sample of size $(n-1)$ from the uniform distribution on the interval $[0, 1]$. Under the alternative, we can choose and fix an arbitrary point on the circle as the zero-direction and cut open the circle at that point to get the line segment $[0, 1]$. Now the $(n-1)$ circular spacings, which do not contain the cut off point, will have the same distribution as $(n-1)$ linear spacings on $[0, 1]$, not containing 0 and 1, while the nth circular spacing, containing the cut off point, will have the same distribution as the sum of the remaining two linear spacings. It is easy to see therefore that the limiting distribution of the empirical distribution function of the normalized or modified spacings is the same in the circular and linear cases. Hence all our statements regarding the ARE's of spacings tests made above hold for the circular case. It is rather fortunate that the circular case, which was our initial interest for this series of investigations on efficiencies, fits into the linear case in the asymptotic theory, and does not create difficult problems as it often does in the case of finite sample sizes.

ACKNOWLEDGEMENTS

The authors wish to thank Professor C. R. Rao for the keen interest he has shown in the work and Professors I. R. Savage and J. K. Ghosh for having gone through an earlier draft of the paper.

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